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Cooperative Games of Choosing Partners and Forming Coalitions in the Marketplace

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Abstract—Two games of interacting between a coalition of players in a marketplace and the residual players acting there are discussed, along with two approaches to fair imputation of gains of coalitions in cooperative games that are based on the concepts of the Shapley vector and core of a cooperative game. In the first game, which is an antagonistic one, the residual players try to minimize the coalition's gain, whereas in the second game, which is a noncooperative one, they try to maximize their own gain as a coalition. A meaningful interpretation of possible relations between gains and Nash equilibrium strategies in both games considered as those played between a coalition of firms and its surrounding in a particular marketplace in the framework of two classes of n-person games is presented. A particular class of games of choosing partners and forming coalitions in which models of firms operating in the marketplace are those with linear constraints and utility functions being sums of linear and bilinear functions of two corresponding vector arguments is analyzed, and a set of maximin problems on polyhedral sets of connected strategies which the problem of choosing a coalition for a particular firm is reducible to are formulated based on the firm models of the considered kind.

Keywords—Antagonistic game, Cooperative game, Core of a cooperative game, Noncooperative game with constant sum, The Shapley vector.

1. INTRODUCTION

It is well known that every firm acting in a marketplace by supplying goods or services (or both) tries to do it in a manner which allows it to gain the best possible (in a particular sense) result or at least to receive its fair share of the market. To this end, such a firm should start with evaluating its potential to compete in the marketplace and the fair share that can be attained in the framework of agreements made with the firm's suppliers, complementers, and possibly, buyers. However, knowing its potential to compete in the marketplace and having made appropriate agreements with the suppliers, complementers, and some of the buyers, the firm should analyze if its guaranteed share of the market welfare can be increased on account of a reasonable cooperation with other firms acting in the same marketplace, and the result of such an analysis should be compared with the share that the firm can secure by only restructuring its own business by any means and acting separately from the other firms. The latter problem

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is mostly associated with identifying the ways of the firm's restructuring and with evaluating the firm's potential for each new structure and can be solved, for instance, using the approach described in [1].

In most instances, the major reason for the firm to seek cooperation with the other firms acting in the same marketplace is dictated by either the firm's intentions to increase its market share or by those to save the existing share which can be decreased if some (or all) of the other firms form a coalition which may act against the firm more effectively than it takes place when all the firms act separately. In the first case, a coalition can be formed by a group of firms acting in the marketplace (that includes the firm) if each of them possesses certain unique values the integrating of which leads to increasing the total share of the coalition acting in the marketplace as a unified player against the others, and the coalition members can agree on a manner of imputing this share increase seeming fair to all of them. In the second case, the firm anticipating unfavorable situations may agree to join a particular coalition even when this act only protects its existing share, whereas the other coalition participants may increase their shares; certainly, this may happen only if there are no other ways of cooperation for the firm that lead to increasing its market share.

It is clear that the firm should analyze benefits to be attained in each potential coalition which it can form or join; if m is the number of firms competing in the same marketplace, the number of potential coalitions that can be formed with the participation of the firm equals 2^{m-1} as such coalitions of two participants can be chosen by $m-1$ ways, those of three participants can be chosen by C_{m-1}^{2} ways, etc., and coalitions of $m-1$ participants can be chosen by C_{m-1}^{m-2} ways. Although the number of coalitions of the above-mentioned kind that should be analyzed by the firm equals 2^{m-1}, for certain purposes, this number can be substantially reduced if, for instance, the firm does not cooperate with certain other firms acting in the marketplace in virtue of strategic of whatever other reasons. Each such a coalition should be considered a unified player acting against the other firms, and it is natural to assume that the coalition members unify their resources (all or some), skills and knowledge to achieve the best possible result (the maximal guaranteed share of the market' welfare), and the best guaranteed result that can be attained by the coalition is that to be attained under the worst possible scenario, which incurs when the firms not joining (or not invited to join) the coalition also unify their potentials (resources, skills, knowledge, etc.) and act as a unified player against the coalition [2]. As well known [2], it is one of concepts that game theory presents for the analysis of coalitions: namely, for each finite number of players (firms competing in the marketplace), for any coalition consisting of K players among T players, another coalition containing remaining $T \setminus K$ players is considered in such a manner that the interaction of the first coalition of K players and the second one of $T \setminus K$ players (this second coalition is sometimes called the surrounding for the first coalition) is viewed as a two-person (Coalition 1 and Coalition 2) zero-sum game, and the best guaranteed result in this game (which may, however, not have a Nash equilibrium (saddle) point) for the first person is considered the best guaranteed result for Coalition 1. If such a guaranteed result (gain of the coalition) can be imputed among the coalition participants in such a manner that a share of the gain to be received by each coalition member under this imputation is not less than the guaranteed result that can be secured by each such a member acting individually in the marketplace against the other firms, the coalition makes sense and may eventually be formed.

Practical implementation of this approach is associated with considering two major problems:

(a) how the firm can evaluate the potential of any coalition to get the market share not less than the sum of shares of its participants acting individually that can be secured by each of them, and

(b) how the participants can work out a fair imputation mechanism to satisfy all the coalition members to an extent making unreasonable the idea of leaving the coalition for any other one.

In this article, we consider both problems and discuss particular mathematical models and approaches that can be employed for analyzing these problems.

2. TWO MODELS OF INTERACTING BETWEEN A COALITION AND ITS SURROUNDING

As known [2,3], manners of forming coalitions and finding fair imputations for members of a particular coalition are studied in the framework of the theory of cooperative games. According to this theory, it is assumed that within a totality of T players each of which has a set of allowable strategies and a utility function, a coalition is formed by a subset of players from this totality in such a manner in which the utility function of the coalition is the sum of utility functions of its members (the assumption on transferability of the utility). The above-mentioned totality of T players, along with their set of allowable strategies and utility functions, can be considered as a (noncooperative) n-person game, and to each subset of the set of T players viewed as a coalition from the totality, a real number equaling the best guaranteed gain for the coalition in the above-mentioned sense is attributed. A function defined on the set of all subsets of the totality of T players whose values coincide with these numbers can be then attributed to the noncooperative game (the characteristic function of the noncooperative game), and coalitions are studied in terms of characteristic functions of corresponding noncooperative games; here, for any $v_G : 2^T \to R^1$, the characteristic function of a noncooperative game of T players (with some sets of allowable strategies and utility functions), the equality $v_G(\emptyset) = 0$ holds [2].

However, the residual players may prefer not to compete with the coalition and, nevertheless, may attain not a less share of the market welfare than that they can secure by counteracting the coalition regardless of their own interests.

Let us consider a noncooperative game G of T players whose utility functions are $H_i(s_1, \dots, s_T)$, $s_i \in S_i$, where S_i is the set of allowable strategies for player i, $i \in \overline{1,T}$, and let v_G be the characteristic function of this game.

Further, let K be a coalition of K players so that $T \setminus K$ is the set of players against which Coalition K competes that form the surrounding for Coalition K if they act as a unified player (coalition). There are two games in the framework which the interaction between Coalition K and Coalition $T \setminus K$ seems reasonable to consider: the following games.

GAME 1.

$$\min_{\eta_{T\setminus K} \in I_{T\setminus K}} \sum_{i\in K} H_i\left(\xi_K, \eta_{T\setminus K}\right) \to \max_{\xi_K \in I_K},$$
$$\max_{\xi_K \in I_K} \sum_{i\in K} H_i\left(\xi_K, \eta_{T\setminus K}\right) \to \min_{\eta_{T\setminus K} \in I_{T\setminus K}},$$

which is an antagonistic game of the coalitions K and $T \setminus K$ with the payoff function

$$\sum_{i\in K} H_i\left(\xi_K, \eta_{T\setminus K}\right).$$

GAME 2.

$$\sum_{i\in K} H_i\left(\xi_K, \eta_{T\setminus K}\right) \to \max_{\xi_K \in I_K},$$
$$\sum_{i\in T\setminus K} H_i\left(\xi_K, \eta_{T\setminus K}\right) \to \max_{\eta_{T\setminus K} \in I_{T\setminus K}},$$

which is a noncooperative game of two players (Coalition K and Coalition $T \setminus K$). Here, it is assumed that each firm has a finite number of (pure) strategies, i.e., both games are finite noncooperative games, and ξ_K and $\eta_{T\setminus K}$ are mixed strategies (probabilistic measures over the sets $I_K = \prod_{i\in K} S_i$, and $I_{T\setminus K} = \prod_{i\in T\setminus K} S_i$, respectively) so that both Game 1 and Game 2 have solutions in mixed strategies according to the Nash theorem on finite noncooperative games [2].

Let $(\xi_K^0, \eta_{T\setminus K}^0)$ and $(\xi_K^*, \eta_{T\setminus K}^*)$ be solutions to Game 1 and Game 2, respectively. Then from the inequality

$$\max_{\xi\in I_K} \sum_{i\in K} H_i(\xi_K, \eta_{T\setminus K}^*) \geq \max_{\xi_K\in I_K} \min_{\eta_{T\setminus K}\in I_{T\setminus K}} \sum_{i\in K} H_i(\xi_K, \eta_{T\setminus K}),$$

one can conclude that

$$\sum_{i\in K} H_i\left(\xi_K^*, \eta_{T\setminus K}^*\right) \geq \sum_{i\in K} H_i\left(\xi_K^0, \eta_{T\setminus K}^0\right).$$

Thus, counteracting Coalition K by the set of firms $T \setminus K$ acting as a coalition (instead of, for instance, trying to get its (Coalition's $T \setminus K$) largest share of the market's welfare) is the worst scenario for Coalition K as in this case, Coalition K cannot guarantee itself a gain exceeding $\sum_{i\in K} H_i(\xi_K^0, \eta_{T\setminus K}^0)$. At the same time, it is not *a priori* clear whether this behavior of Coalition $T \setminus K$ may be less reasonable in terms of the gain that can be attained than the one associated with pursuing its own best result (the largest share of the market) while Coalition K is in pursuit of its largest share of the same market. It turns, however, out that for certain classes of noncooperative games, the guaranteed gain for Coalition $T \setminus K$ in Game 1 and the equilibrium gain for coalition $T \setminus K$ in Game 2 coincide; in particular, it is true for noncooperative n-person games with constant sum, where there exists a certain volume of the market's welfare W that remains the same for each set of allowable strategies of the players [1]. Namely, let v_G be the characteristic function of a (noncooperative) game of T players with constant sum so that the relations

$$v_G(T) = W = \sum_{i\in T} H_i(s), \qquad s \in \prod_{i=1}^{T} S_i$$

hold, along with the equality

$$v_G(K) + v_G(T \setminus K) = v_G(T)$$

for any coalition K of the totality of T players [2].

THEOREM. *(See [4].) The pair $(\xi_K^0, \eta_{T\setminus K}^0)$ is a solution to Game 1 if and only if this pair is a solution to Game 2.*

COROLLARY. *(See [4].) For any solution $(\xi_K^0, \eta_{T\setminus K}^0)$ to Game 1 and any solution $(\xi_K^*, \eta_{T\setminus K}^*)$ to Game 2, the equality*

$$\sum_{i\in T\setminus K} H_i\left(\xi_K^*, \eta_{T\setminus K}^*\right) = \sum_{i\in T\setminus K} H_i\left(\xi_K^0, \eta_{T\setminus K}^0\right)$$

holds.

At the same time, as shown in [5], all three relations $<$, $>$, and $=$ may hold between the numbers $\sum_{i\in T\setminus K} H_i(\xi_K^*, \eta_{T\setminus K}^*)$ and $\sum_{i\in T\setminus K} H_i(\xi_K^0, \eta_{T\setminus K}^0)$ in the general case, when the corresponding T-person noncooperative games are not games with constant sum. This, in particular, means that the surrounding (Coalition $T \setminus K$) should analyze the option to play Game 2 rather than counteract Coalition K, regardlessly.

3. CHOOSING A COALITION BY A FIRM

As mentioned before, considering the worst scenario for each Coalition K which the firm may form or join is reducible to solving a number of antagonistic (zero-sum) games, and the calculating of $v_G(K)$ is the calculating of values of these games; however, for certain functions $H_i(s)$, solving such games may present substantial difficulties. It is important to emphasize that solving these games allows the firm finding a set of coalitions which the firm should consider as perspective (if this set is nonempty), i.e., those capable, in principle, of bringing a better piece of the market's welfare to the firm than the one that the firm can secure acting separately against the other market participants. It is this set of coalitions which the firm will analyze in terms of manners of imputing the guaranteed coalition gain for each coalition from the set, and members of these coalitions are the firms with which it (the firm) may consider to negotiate the corresponding imputations (within corresponding coalitions).

Let us consider T firms competing in the same marketplace for each of which volumes of m products to be supplied to the marketplace are described by the vector $A^i x^i$, and let $x^i \in M^i$, where M^i is a polyhedron given by a compatible system of linear inequalities, $i \in \overline{1,T}$; such mathematical models are widespread, for instance, in industrial and transportation systems and were considered, in particular, in [1]. Let further h^i be a vector whose components are as such that the function $\langle h^i, x^i \rangle$ describes expenditures that firm i bears producing and supplying the products to the marketplace in volumes corresponding to components of the vector $A^i x^i$; to simplify the notation, we further assume that all the firms supply the same kinds of products to the marketplace. Let, finally, K be a coalition consisting of K firms, say, those with numbers one through K, for the sake of definiteness, and let the marketplace be characterized by a vector of the expected demand d for all the products in such a manner that

$$\sum_{i=1}^{T} A^i x^i = d, \qquad d \in S = \left\{ d \in R_+^m : Dd \leq \delta \right\},$$

where D and δ are a matrix and a vector of corresponding dimensions, as well as by the vector of the expected amounts of money l to be available in the marketplace for each product, along with its total amount, so that $l \in L = \{ l \in R_+^m : Gl \leq \alpha \}$, where G and α are a matrix and a vector of corresponding dimensions, and it is natural to assume that both S and L are polyhedra (bounded polyhedral sets).

Let p^i be the vector whose components are prices at which firm i supplies its products to the marketplace; it is obvious that for each product $j, j \in \overline{1,m}$, for each firm i, the system of linear inequalities $\underline{p}_j^i \leq p_j^i \leq \overline{p}_j^i$ should hold, where $\underline{p}_j^i, \overline{p}_j^i$ are certain real positive numbers. Such numbers could be assigned by experts or determined by one way or the other; for instance, one can easily conclude that

$$\overline{p}_j^i \leq \max_{l \in L,\ d \in S} \frac{\langle (e_j, 0), (l, d) \rangle}{\langle (0, e_j), (l, d) \rangle}, \qquad j \in \overline{1,m}, \quad i \in \overline{1,T},$$

where $e_j \in R^m$ is the vector all whose components equal zero, except for component j equaling one, and the problem of finding the maximum in the right-hand side of the inequality is a fractional-linear programming problem for which various solving methods, along with the software implementing these methods, exist.

The numbers $\underline{p}_j^i$ are determined by each firm i proceeding from the ratio between the revenue and expenditures that the firm can afford, which is described by a fractional-quadratic function

$$\frac{\langle h^i, x^i \rangle}{\langle p^i, A^i x^i \rangle} = \varphi^i \left(p^i, x^i \right).$$

It seems natural to assume that the inequalities

$$p^i \geq \alpha^i > 0, \qquad i \in \overline{1,T},$$

and

$$\left\langle p^i, A^i x^i \right\rangle > 0, \qquad i \in \overline{1,T},$$

hold; here, $\alpha^i, i \in \overline{1,T}$ are certain vectors of prices of the products components of which may be taken, for instance, as the lowest prices observed in any statistical data available to the firm, and the second set of inequalities mean that each firm does not supply all its products to the marketplace free of charge.

Certainly, each firm has its own view on what values of $\varphi^i(p^i, x^i)$ are acceptable to it, which means that the numbers β_i such that the inequalities

$$\varphi^i\left(p^i, x^i\right) \leq \beta_i, \qquad i \in \overline{1,T},$$

hold are known.

While firm i can choose the vector $x^i \in M^i$ in order to reduce the value of the function $\varphi^i(p^i, x^i)$, the analysis of the worst scenario for the firm is associated with finding

$$\min_{x^i \in M^i} \max_{p^i \geq \alpha^i} \frac{\langle h^i, x^i \rangle}{\langle p^i, A^i x^i \rangle},$$

and the vector $\tilde{p}^i$ in the pair of vectors $(\tilde{p}^i, \tilde{x}^i)$ for which

$$\varphi^i\left(p^i, x^i\right) \leq \varphi^i\left(\tilde{p}^i, \tilde{x}^i\right) = \beta_i = \min_{x^i \in M^i} \max_{p^i \geq \alpha^i} \frac{\langle h^i, x^i \rangle}{\langle p^i, A^i x^i \rangle}$$

can be viewed as the one components of which determine the lowest prices securing the ratio level not exceeding the one defined by the number β_i so that $\underline{p}^i_j = \tilde{p}^i_j$, $j \in \overline{1,m}$, $i \in \overline{1,T}$ can be considered acceptable.

For the coalition of K firms under consideration, its utility function (assuming that all the firms forming the coalition unify their efforts in order to increase their cumulative profit) is described by the function

$$\sum_{i=1}^{K} \left\{\langle p^i, A^i x^i \rangle - \langle h^i, x^i \rangle\right\},$$

and the best guaranteed result (profit) that the coalition can secure equals

$$\max_{(x^i, p^i),\ i \in \overline{1,K}} \ \min_{(x^\tau, p^\tau),\ \tau \in \overline{K+1,T}} \sum_{i=1}^{K} \left\{\langle p^i, A^i x^i \rangle - \langle h^i, x^i \rangle\right\},$$

where: $x^i \in \tilde{M}^i$, and polyhedra $\tilde{M}^i$ either coincide with M^i or are those of the kind

$$M^i \cap \left\{x^i \geq 0 : \langle h^i, x^i \rangle \leq \beta^i \langle \tilde{p}^i, A^i x^i \rangle\right\}, \qquad i \in \overline{1,T};$$

vectors $\underline{p}^i$, $i \in \overline{1,T}$, are solutions to the problems

$$\max_{p^i \geq \alpha^i} \frac{\langle h^i, x^i \rangle}{\langle p^i, A^i x^i \rangle} \to \min_{x^i \in M^i},$$

(which can be solved, in particular, by well-known methods of minimax optimization [6]) or are assigned by experts; components of the vector $\overline{p}^i$ satisfy linear inequalities

$$\overline{p}^i_j \leq \overline{p}^\tau_j, \qquad \tau \in \overline{K+1,T}, \quad i \in \overline{1,K},$$

where

$$\overline{p}^\tau_j \leq \max_{l \in L, d \in S} \frac{\langle (e_j, 0), (l, d) \rangle}{\langle (0, e_j), (l, d) \rangle}, \qquad j \in \overline{1,m}, \quad \tau \in \overline{K+1,T};$$

and pairs of vectors $(x^i, p^i), (x^\tau, p^\tau) : i \in \overline{1, K},\ \tau \in \overline{K+1, T}$ also satisfy the system of equalities and inequalities

$$\sum_{i=1}^{T} A^i x^i = d,$$
$$Dd \le \delta, \qquad d \in R_+^m.$$

To 'make the further presentation more observable, let $z = (x^1, \dots, x^K)$, $y = (x^{K+1}, \dots, x^T)$, $u = (p^1, \dots, p^K)$, $v = (p^{K+1}, \dots, p^T)$. Then the system of linear inequalities for (x^i, p^i), $i \in \overline{1, K}$ and (x^τ, p^τ), $\tau \in \overline{K+1, T}$ can be rewritten in the vector-matrix form

$$Az + By \le \omega,$$
$$Cu + Hv \le \gamma,$$

whereas

$$\sum_{i=1}^{K} \left\{ \langle p^i, A^i x^i \rangle - \langle h^i, x^i \rangle \right\} = \langle u, Qz \rangle - \langle q, z \rangle,$$

where A, B, C, H, Q are matrices of corresponding dimensions assembled from elements of the matrices A^i, D and from those participating in the description of M^i, $i \in \overline{1, n}$ and h^i, $i \in \overline{1, K}$, as well as from the numbers $+1$, -1, and β^i, $i \in \overline{1, K}$, whereas ω, γ, q are vectors of corresponding dimensions assembled from components of the vectors $d, \delta, \underline{p}^i, \overline{p}^i$, along with components of vectors participating in the description of M^i, $i \in \overline{1, T}$ and; examples of forming matrices of such a kind from systems of linear equalities can be found, in particular, in [1].

The problem of finding the best guaranteed result for Coalition K can be rewritten as

$$\min_{(y,v)} \left\{ \langle u, Qz \rangle - \langle q, z \rangle \right\} \to \max_{(z,u)},$$
$$Az + By \le \omega,$$
$$Cu + Hv \le \gamma,$$

and it is a maximin problem on polyhedral sets of connected strategies; an approach to solving such problems was proposed in [7].

It is important, however, to emphasize that for each Coalition K, the corresponding system contains a different number of inequalities reflecting competitiveness in prices for the coalition members for the same products; it leads, in turn, to different polyhedra for the vector (u, v) and, correspondingly, different matrices C and H to be considered for each Coalition K.

Thus, for a particular problem of choosing a coalition under study, a number of maximin problem on polyhedral sets of connected strategies of the considered kind should be solved. Certainly, this assertion remains true if not all the firms produce and supply all the products to the marketplace; corresponding changes in the proposed model can be easily made in such cases.

4. FINDING A FAIR IMPUTATION OF THE COALITION'S GAIN

In this section, we discuss two basic approaches to determining imputations of the gain attained by a coalition, namely, calculating the game core and calculating the Shapley vector, and a particular relation, namely, the membership relation, between the Shapley vector and the core of a cooperative game. To this end, we first, briefly remind these two concepts in conformity to a set of firms acting in a marketplace using the same terminology that was employed in previous sections of this article and adhering to a scheme of the concepts presentation close to the one used in [2]. One should, however, bear in mind that although finding a fair imputation of a coalition gain is a part of the approach to choosing partners and forming coalitions by the firm acting in the marketplace, which is the subject of this article and in virtue of that, deserves to be spotlighted here to a ceratin extent, no new mathematical models are presented and no new mathematical results are developed in this section of the article, which, in fact, possesses only a reference character, and calculations presented further in this section are only illustrative of that the above-mentioned relation may both hold and not hold in particular games.

The problem of finding a fair imputation of $v_G(K)$ among members of a coalition of K participants is the second problem mentioned in Section 1 of this article that the firm faces while analyzing the ways to increase its market share, and an imputation mechanism substantially affects the possibility of creating a coalition even when such a coalition can give more to each of its participants than each participant can get acting separately in the marketplace. However, first of all, the firm should understand how much of the market's welfare is available to all the firms competing in the marketplace, regardless of what coalitions they may want to form, and whether there exists a reasonable imputation of this amount of welfare among all the firms that could be more attractive to them than any imputation within any coalition that the firm can offer to them to form with its own participation.

It is clear that if all T firms had acted in the marketplace as a unified player combining all their skills, knowledge, resources, etc., they would have attained the amount of the market's welfare equaling

$$v_G(T) = \max_{s \in \prod_{i \in T} S_i} \sum_{i \in T} H_i(s)$$

so that the amount of the market's welfare to be imputed among firms forming a particular Coalition K does not exceed $v_G(T)$. As known, in cooperative game theory, a vector $x = (x_1, \dots, x_T) \in R_+^T$, where T is the number of players (in our case, firms), is called an imputation in T-person game with the characteristic function v_G if its components satisfy the following two relations [2,3]: $x_i \geq v_G(\{i\})$ (personal rationality) and $\sum_{i \in T} x_i = v_G(T)$ (collective rationality), $i \in \overline{1,T}$. For a coalition of K players, it is said that the imputation $(x_1, \dots, x_T)$ dominates the imputation $(y_1, \dots, y_T)$ with respect to coalition K if $y_i < x_i$, $i \in K$ and $\sum_{i \in K} x_i \leq v_G(K)$, which means that Coalition K prefers the imputation x to the imputation y. Thus, it is easy to conclude that the imputation $x = (x_1, \dots, x_T)$ is not dominated with respect to any coalition that may be formed out of T players in the game with the characteristic function $v_G(T)$ and contains more than one player and not more than $T - 1$ players if and only if $v_G(K) \leq \sum_{i \in K} x_i$ for any $K \subset T$ [2,3]; a set of all nondominated imputations in a cooperative game is called the core of the game.

If this set (the game core) is nonempty (for certain games with constant sum, namely, for those whose characteristic functions are not additive, this set turns out to be empty [2]), then no coalition can give its members more than they can get according to the imputation from the game core as the best guaranteed gain for the coalition cannot exceed the amount of the market's welfare to be gotten by the coalition as a result of this imputation, and the imputation from the game core is considered stable in this sense; it is also obvious that if the core of a cooperative game is not empty, it is a polyhedral set.

If the game core is empty, then the firm can order the coalitions in which it participates in the value of v_G and for each Coalition K from this ordered set, can calculate the number $v_G(K) - \sum_{i \in K} v(\{i\})$; for those coalitions for which this difference is positive, the firm may further negotiate with the participants a manner of imputing this difference among them (for instance, proportionally to $v(\{i\})$). Negotiations on this matter may end up successfully if there exists such a Coalition K that for each firm from this coalition, imputations within other coalitions which this firm can participate in give it the gain not exceeding the one it gets by participating in Coalition K.

Another approach to imputing the market's welfare among the players (firms acting in the marketplace) is associated with a special game-theoretical construction, namely, the Shapley vector [2,8], which provides an imputation of the market's welfare, regardless of the emptiness or nonemptiness of the game core and is considered fair in a certain sense. It is the concept of fairness that determines requirements to an imputation in a cooperative game and serves as grounds for the Shapley vector construction. According to this concept, among all players in a cooperative game, those who do not bring to any coalition an amount of the welfare exceeding the amount that each of them can gain acting separately against all the other players, called dummies, are distinguished, and the rest of the players, called the support of the game, is that very subset of the set of players among which the imputation of a part of the market's welfare attained by a coalition is done. Namely, it is considered fair that nothing is taken from and is given to dummies in the game, and

$$v_G(T) - \sum_{j \in D_G} v_G(j),$$

where D_G is the set of dummies in the game, is the amount of the market's welfare that is divided among the support of the game.

It is further considered fair that for each coalition, there exist a certain order of positions in the coalition (which can be viewed, for instance, as the order in which firms agree to join the coalition in the course of negotiations that the firm may hold in pursuit of forming a particular coalition) such that these positions are rewarded in the course of imputing the market's welfare in the sense that the amount of the reward does not depend on what particular player occupies a particular position in the coalition. Finally, it is considered fair that for one and the same set of players participating in two games (with two different characteristic functions) concurrently, the gain attained by any coalition of players equals the sum of gains determined by the characteristic function of each game for this coalition (and, in particular, for each player) [8].

It turns out that for each cooperative game G with m players and characteristic function v_G, there exists the only vector $\varphi(v_G)$ satisfying these three principles of fairness and being an imputation, i.e., satisfying conditions of personal and collective rationalities, and coordinates of this vector are determined as follows:

$$\varphi_i(v_G) = \sum_{Q \ni i} \frac{(m - |Q|)!(|Q| - 1)!}{m!} \left(v_G(Q) - v_G(Q \setminus \{i\})\right), \qquad i \in \overline{1, m},$$

where m is the number of players in the game, Q is a coalition in the game containing Player i, $Q \setminus \{i\}$ is the coalition consisting of all players from Q, except for i, and $|Q|$ is the number of elements in the set Q. One can easily figure out that $(v_G(Q) - v_G(Q \setminus \{i\}),\ i \in \overline{1, m})$ represents the value that Player i contributes to Coalition Q, whereas $(m - |Q|)!(|Q| - 1)!$ equals the number of permutations in which Player i occupies Position i in the set of positions occupied by all players in the game. If orders in which players occupy their positions in the set of m positions are equally likely to occur, than $(m - |Q|)!(|Q| - 1)!/m!$ equals the probability that Player i occupies Position i in the set, and $\varphi_i(v_G)$ is the expectation of the gain increment associated with joining a coalition (which does not include Player i) by Player i [2,8].

A simple arithmetic consideration shows that, in particular in the game with T players (firms) under consideration, $\varphi_i(v_G) = v(\{i\})$ for each dummy in the game, $\sum_{i \in T} \varphi_i(v_G) = v(T)$, and $\sum_{i \in S} \varphi_i(v_G) = v(S)$ for $S \subset T$, where S is the support of the game, and that $\varphi_i(v_G) \geq v(\{i\})$ for each player in the game [2].

It is substantial to emphasize that the Shapley vector is determined explicitly, whereas vectors from the game core are determined implicitly (as a set of solutions to a system of linear inequalities), and that the Shapley vector always exists in any cooperative game, whereas the game core may be empty. As choosing a vector from the (nonempty) core of a game and calculating the Shapley vector are two basic approaches to the imputation, it seems expedient to find what relations may exist between two imputations calculated according to these approaches. It turns out that if the game core of a cooperative game is not empty, the Shapley vector does not necessarily belong to this core, i.e., may not necessarily be a solution to the system of linear inequalities determining the game core which is clear from the following example.

EXAMPLE. Let G be a general cooperative game with three players in the $0-1$ reduced form [2], and v_G be the characteristic function of the game such that

$$v_G(\emptyset) = v_G(1) = v_G(2) = v_G(3) = 0,$$
$$v_G(1,2,3) = 1, \qquad v_G(1,2) = c_3, \qquad v_G(1,3) = c_2, \qquad v_G(2,3) = c_1,$$

where $0 \le c_i \le 1$, $i \in \overline{1,3}$, considered, in particular, in [2]. For this game, the Shapley vector is determined by the following system of linear inequalities:

$$\begin{aligned} \varphi_1(v_G) &= \frac{1}{6}c_2 + \frac{1}{6}c_3 + \frac{1-c_1}{3}, \\ \varphi_2(v_G) &= \frac{1-c_2}{3} + \frac{1}{6}c_3 + \frac{1}{6}c_1, \\ \varphi_3(v_G) &= \frac{1}{6}c_1 + \frac{1}{6}c_2 + \frac{1-c_3}{3}, \end{aligned}$$

as, for instance, there are three coalitions that contain Player 1, and for these three coalitions, the equalities

$$v(1,2) - v(2) = c_3, \qquad v(1,3) - v(3) = c_2, \qquad v(1,2,3) - v(2,3) = 1 - c_1$$

hold, along with the equalities

$$\begin{aligned} \frac{(3-|(1,2)|)!(|(1,2)|-1)!}{3!} &= \frac{1! \times 1!}{3!} = \frac{1}{6}, \\ \frac{(3-|(1,3)|)!(|(1,3)|-1)!}{3!} &= \frac{1! \times 1!}{3!} = \frac{1}{6}, \\ \frac{(3-|(1,2,3)|)!(|(1,2,3)|-1)!}{3!} &= \frac{2!}{3!} = \frac{1}{3}, \end{aligned}$$

so that

$$\varphi_1(v_G) = \frac{1}{6}c_2 + \frac{1}{6}c_3 + \frac{1-c_1}{3},$$

and the formulae for $\varphi_2(v_G)$ and $\varphi_3(v_G)$ can be obtained analogously.

At the same time, the game core is determined by the following system of linear inequalities:

$$\begin{aligned} x_1 + x_2 &\ge v_G(1,2) = c_3, \\ x_1 + x_3 &\ge v_G(1,3) = c_2, \\ x_2 + x_3 &\ge v_G(2,3) = c_1, \end{aligned}$$

which is equal to the system

$$\begin{aligned} x_1 &\le 1 - c_1, \\ x_2 &\le 1 - c_2, \\ x_3 &\le 1 - c_3, \end{aligned}$$

as $x_1 + x_2 + x_3 = 1$ according to the condition of collective rationality that holds for every imputation (x_1, x_2, x_3); from the same condition, it stems that if the latter system has a feasible solution, i.e., the game core is not empty, then

$$c_1 + c_2 + c_3 \le 2.$$

Substituting expressions for components of the Shapley vector into the system of linear inequalities determining the game core, one can be easily certain that the Shapley vector $(\varphi_1(v_G), \varphi_2(v_G), \varphi_3(v_G))$ belongs to the game core if and only if the system of linear inequalities

$$\frac{1}{6}c_2 + \frac{1}{6}c_3 + \frac{1}{3} - \frac{1}{3}c_1 + \frac{1}{3} - \frac{1}{3}c_2 + \frac{1}{6}c_3 + \frac{1}{6}c_1 \geq c_3,$$
$$\frac{1}{6}c_2 + \frac{1}{6}c_3 + \frac{1}{3} - \frac{1}{3}c_1 + \frac{1}{6}c_1 + \frac{1}{6}c_2 + \frac{1}{3} - \frac{1}{3}c_3 \geq c_2,$$
$$\frac{1}{3} - \frac{1}{3}c_2 + \frac{1}{6}c_3 + \frac{1}{6}c_1 + \frac{1}{6}c_1 + \frac{1}{6}c_2 + \frac{1}{3} - \frac{1}{3}c_3 \geq c_1,$$

holds. This system can be rewritten in the form

$$c_1 + c_2 + 4c_3 \leq 4,$$
$$c_1 + 4c_2 + c_3 \leq 4,$$
$$4c_1 + c_2 + c_3 \leq 4,$$

and the inequality

$$c_1 + c_2 + c_3 \leq 2$$

is also a corollary from the latter system of linear inequalities. It is easy to conclude that, for instance, for $c_1 = 1/2$, $c_2 = 1/2$, $c_3 = 1/2$, this system holds so that the Shapley vector (1/3, 1/3, 1/3) of the game for which $v_G(1,2) = v_G(1,3) = v_G(2,3) = 1/2$ belongs to the core of this game determined by the system of linear inequalities

$$x_1 + x_2 \geq \frac{1}{2},$$
$$x_1 + x_3 \geq \frac{1}{2},$$
$$x_2 + x_3 \geq \frac{1}{2},$$

whereas the Shapley vector $(1/6, 1/6, 2/3)$ of the game for which

$$v_G(1,2) = 0 = c_3, \qquad v_G(1,3) = c_2 = v_G(2,3) = c_1 = 1$$

does not belong to the core of the game determined by the system of linear inequalities

$$x_1 + x_2 \geq 0,$$
$$x_1 + x_3 \geq 1,$$
$$x_2 + x_3 \geq 1,$$

where in both games, their cores are not empty.

One should emphasize that the analysis of whether the Shapley vector belongs to the nonempty core of a particular cooperative game implies calculating components of this vector and verifying whether these components considered as values of corresponding variables, i.e., $x_i = \varphi_i(v_G)$, $i \in \overline{1,T}$ in the game with T players, satisfy linear inequalities describing the core of the game. Calculations presented in the Example are only illustrative of this general scheme while necessary and sufficient conditions for the Shapley vector to belong to the (nonempty) core of a three person cooperative game, along with formulae for calculating components of this vector in such a game, can be found in text books on game theory (in particular, in the second edition of the book [2] available, however, only in Russian) although they are a direct corollary (as was demonstrated in the Example) from formulae for the Shapley vector and the core of a cooperative game. Certainly, similar formulae can be obtained for cooperative games with more than three players in just the same way it was done for the three person game in the Example once the above-mentioned analysis is conducted for such games; however, the consideration of three person games is sufficient to make a conclusion on the membership relation between the Shapley vector and the core of a cooperative game with T players.

5. CONCLUSION

1. Choosing partners to form a coalition in the marketplace necessitates solving a set of antagonistic games, and a number of problems to be solved may turn out to be rather large.
2. The fairness of the imputations should be understood in one and the same sense by partners forming a coalition and, generally, may not be attained in the framework of either concept of fairness considered in this article; other concepts of fairness satisfactory for a particular marketplace and a particular set of players (firms acting in the marketplace) can be developed although they may not possess certain mathematical properties as do vectors from the core or the Shapley vector while being acceptable to the players. At the same time, the Shapley vector from the core of a particular cooperative game represents an imputation that may seem reasonable to the coalition members as a nondominated, stable imputation, which is also fair in the sense under consideration.
3. Recommendations to be obtained by a firm as a result of solving problems associated with choosing partners to form a coalition and finding a fair (in any particular sense) imputation of the gain that can be attained by the coalition should be considered only as preliminary evaluations of the firm's ability to increase its share of the market's welfare. Certain limitations on forming coalitions imposed, first of all, by legal regulations within which the market should function in a particular location may substantially affect both sets of available strategies for the market participants and the size of the welfare to be divided among them.
4. The reasoning presented in Section 3 of this article implies that calculated or assigned vectors $\underline{p}^i$ are as such that the inequalities

$$\underline{p}^i \leq \overline{p}^i, \qquad i \in \overline{1,T},$$

hold.

If

$$\underline{p}^i_j = \overline{p}^i_j,$$

for any $i \in \overline{1,T}$, $j \in \overline{1,m}$, then corresponding components of the vectors u or (and) v are these constants.

If, however, for any $i \in \overline{1,T}$ and $j \in \overline{1,m}$, the number $\underline{p}^i_j$ found as, for instance, proposed in Section 3 of the article turns out to be as such that the inequality

$$\underline{p}^i_j > \overline{p}^i_j, \qquad i \in \overline{1,T},$$

holds, then product j produced by firm i is not competitive in the marketplace.
5. The approach to choosing partners and forming coalitions considered in this article is a simplified one although it allows the firm to evaluate strategies of possible enlargements of the firm's market share by solving mathematical programming problems. More general approaches can be developed based on advanced results of cooperative game theory [3] once it is dictated by any particular strategic management problems that a particular firm faces in a particular marketplace.

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