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Incentive contracts in delegated portfolio management under VaR constraint ☆

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ABSTRACT

This paper studies the incentive effect of linear performance-adjusted contracts in delegated portfolio management under a value-at-risk (VaR) constraint. It is shown that a linear performance-based contract can provide incentives for the portfolio manager to work at acquiring private information under a VaR risk constraint. The expected utility and optimal effort of a risk-averse manager are increasing functions of the return sharing ratio in the contract. However, a risk constraint causes the portfolio manager to reduce effort in gathering private information, suggesting that the VaR constraint increases the moral hazard between the investor and the manager.

1. Introduction

In the past few decades, the industry of delegated portfolio management has developed tremendously. Investors delegate professional investment managers to manage their wealth believing that portfolio managers possess the ability to collect, interpret, and apply profitably the information on risk and return of financial assets. However, because managerial ability and effort are not observable, investors have to design an appropriate compensation contract to induce managers to work hard to acquire better information in the market so as to reduce the agency cost.

Although the performance-based linear incentive contract has been widely used in the delegated portfolio management industry, its incentive effect has not been fully developed theoretically, especially under various market or institutional complications. Stoughton (1993) and Admati and Pleiderer (1997) find that a linear contract cannot induce managers to work hard to acquire private information, which is the well-known “no-incentive” feature of linear contract under ideal market conditions. In addition, Admati and Pleiderer (1997) argue that the commonly used benchmark-adjusted compensation schemes are generally inconsistent with optimal risk-sharing and do not lead to the choice of an optimal portfolio for the investor.

This paper studies the linear incentive contract under a value-at-risk (VaR) constraint in a framework similar to Stoughton (1993) and Gomez and Sharma (2006). As Jorion (2003, 2007) and Pearson (2002) note, the fund management industry is increasingly using VaR to: (1) allocate assets among managers, (2) set risk limits, and (3) monitor asset allocations and managers (these activities are often referred to as ‘risk budgeting’). However, the incentive effect of linear contracts of delegated portfolio management under a VaR constraint has not been investigated in the literature. In this paper we show that a linear contract with a VaR risk constraint can raise the manager's efforts on information acquisition. A risk-averse manager's expected utility and optimal effort levels increase with the return sharing ratio, suggesting that a linear contract can not only allocate risks efficiently between the investor and the manager, but also induce the manager to work hard. Furthermore, under a VaR constraint, the manager's effort level is lower than that without such a constraint, suggesting that the VaR constraint increases the moral hazard between the investor and the manager.

1.1. Literature review

Despite the somewhat surprising result in Stoughton (1993), the linear contract is still the most popular contract in asset management

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due to its simplicity and intuitiveness.[1] Two strands of literature have addressed the no-incentive feature of linear contracts.[2] The first strand studies nonlinear incentive contract and the second strand investigates the impact of various practical complications or restrictions (some of which are discussed in Almazan et al., 2004) on the incentive effect of linear contracts. The general finding of these studies is that the practical complications (e.g., restriction on short-selling or the investment manager's possession of market power) can usually overcome the underinvestment of effort in the linear contract.

In the first strand of literature, Bhattacharya and Pleiderer (1985) prove the existence of incentive contracts that can distinguish the types of managers and make the managers to truthfully reveal their ability and private information. In such a contract the compensation of the manager should be a quadratic function of the asset return. A nonlinear contract is optimal for investors with high risk tolerance. Ross (2004) examines the incentive effects of some common structures such as puts and calls, and briefly explores the duality between a fee schedule that makes an agent more or less risk-averse, and gambles that increase or decrease risk. Li and Tiwari (2009) argue that a benchmark-adjusted option-type incentive can help overcome the problem of underinvestment in effort that undermines linear contracts.

In the second strand of related literature, Gomez and Sharma (2006) and Agarwal et al. (2007) show that under short-sell constraints a linear contract can provide portfolio managers with incentives to gather information. Sheng and Yang (2010) demonstrate that when the manager has market power, that is, her asset selection can influence the equilibrium market price, a linear contract has incentive effects. Dybvig et al. (2010) show that trading restrictions are an essential part of an optimal contract because they prevent the manager from undoing the incentive effects of performance-based fees. Kyle et al. (2011) endogenize information acquisition and portfolio delegation in a one-period strategic trading model, and find that a higher powered linear contract induces the manager to exert more effort for information acquisition. Fabretti and Herzel (2012) consider the problem of how to establish compensation for a portfolio manager who is required to restrict the investment set because of socially responsible screening. Palomino and Prat (2003) show that when an agent can control the riskiness of the portfolio in a two-period model, the optimal contract is simply a bonus contract: the agent is paid a fixed sum if the portfolio return is above a certain threshold; in a multi-period framework, the optimal contract is linear. Heinkel and Stoughton (1994) also investigate the dynamics of portfolio management contracts in a two-period model.

In delegated portfolio management, the manager wants to maximize her expected utility, but preferences on risk taking may differ between the investor and the manager. To protect her interest the investor may want to introduce risk constraints into the delegated portfolio management contract. Risk constraints can change investing opportunities and behavior of the manager. Jorion (2003) finds that adding a constraint on the total portfolio volatility can substantially improve the performance of an active portfolio. In a mean tracking error variance (TEV) framework, Alexander and Baptista (2008) find that a VaR constraint can mitigate the well-known problem that when an active manager seeks to track (or outperform) a benchmark, she may select a particularly inefficient portfolio.

This paper falls into the second strand of literature by studying the incentive effect of a linear contract under a VaR risk constraint.

[1] An institutional reason for the popularity of linear contracts is that SEC restricts compensation contracts in the mutual fund industry to only linear symmetric contracts. As reported in Ma et al. (2012), over ninety-five percent of portfolio managers have salary-plus-bonus type of compensation contracts in the U.S. mutual fund industry.

[2] See Stracca (2006) for a review of theoretical literature on delegated portfolio management.

1.2. VaR in financial risk management

The origin of VaR can be traced back to quantitative trading groups at several financial institutions in the 1980s, notably Bankers Trust, although neither the acronym "VaR" nor the definition were standardized. When the financial events of the early 1990s called for some measure on firm-wide risk, VaR was the natural choice because it was the only common risk measure that could be both defined for all businesses and aggregated without strong assumptions. In 1994, J. P. Morgan published the methodology, which, two years later, was spun off into an independent for-profit business now part of RiskMetrics Group.

In 1997, the U.S. Securities and Exchange Commission ruled that public corporations must disclose quantitative information about their derivatives activity. Major banks chose to implement the rule by including VaR information in the notes of their financial statements. Worldwide adoption of the Basel Accords since late 1990s gave further impetus to the use of VaR. In the meantime, many academic researches have studied its application and impact in portfolio management (e.g., Basak and Shapiro, 2001; Campbell, et al., 2001; Jarrow and Zhao, 2006; Natarajan, et al., 2008).

The current paper proceeds as follows. Section 2 describes the basic model employed. Section 3 outlines the results of Stoughton (1993) and Gomez and Sharma (2006), used as a benchmark for our findings, which propose that there is no incentive effect in a linear contract. The VaR constraint is introduced in Section 4. Section 5 studies the manager's utility function with a risk constraint. Section 6 investigates the impact of a VaR constraint on the incentive effect in a linear contract. Concluding remarks are in Section 7.

2. The model

The basic setup of the model is very similar to that in Stoughton (1993) and Gomez and Sharma (2006). In an economy there are two representative agents, an investor and an investment manager. There are two types of assets on the market, a risky asset and a riskless asset. The expected return of the riskless asset is zero. The manager receives a private signal about the return of the risky asset, $\tilde{y}=\tilde{x}+\tilde{z}$, where $\tilde{x}$ is the true rate of the return of the risky asset and $\tilde{z}$ represents a noise term that is uncorrelated with $\tilde{x}$. We assume that $\tilde{x}$ is of a standard normal distribution $\tilde{x}\sim N(0,1)$.[3] Let $\tilde{z}\sim N\left(0,\sigma_z^2\right)$, with $\sigma_z^2<\infty$, such that a higher σ_z^2 implies a less precise signal, which in turn is caused by a lower level of effort by the manager to acquire the signal. Suppose that ρ represents the effort level of the manager, so $\sigma_z^2=\rho^{-1}$ reflects the manager's effort level. As per Stoughton (1993), the signal's precision is $\rho/(1+\rho)$, which is an increasing and concave function of the effort level. When the manager receives her private signal, $\tilde{y}=y$, she updates her expectation on the return of the risky asset by the Bayesian rule. Now its conditional expectation and variance are $E(\tilde{x}|y)=\frac{\rho}{1+\rho}y$ and $\mathrm{Var}(\tilde{x}|y)=\frac{1}{1+\rho}$ respectively, that is, $\tilde{x}|y\sim N\left(\frac{\rho}{1+\rho}y,\frac{1}{1+\rho}\right)$. The manager is a price taker, so her behavior does not affect the equilibrium market price. In her portfolio, the proportions invested in the risky and the riskless assets are $\theta(y)$ and $1-\theta(y)$ respectively.

The investor delegates the investment decision-making power to the manager, and provides a linear fee schedule $\beta_0+\beta\tilde{W}$ to the manager, with $\beta_0,\beta>0$ as contract parameters. $\tilde{W}=\theta(y)\tilde{x}$ represents the return of the portfolio at the end of the period. For not losing generality, we assume that the initial amount of investment is one.

Similar to Stoughton (1993), we assume the cost of effort of the manager is $V(r,\rho)$, where r is the absolute risk aversion coefficient, $V'_\rho(r,\rho)>0$, $V''_{\rho\rho}(r,\rho)\ge 0$, $V(r,0)=0$, and $V'_r(r,\rho)\ge 0$. $V(r,\rho)$ represents

[3] The results do not change if the expected return is not zero. Moreover, none of our results depends, qualitatively, on the assumptions of normal distribution.

disutility caused by effort. At the same effort level, the higher the risk aversion, the higher the disutility.

According to Gomez and Sharma (2006), to assure the existence of the optimal effort level of the manager, we assume $\frac{V''_{\rho\rho}(r,\rho)\rho}{V'_{\rho}(r,\rho)} > \frac{\rho}{1+\rho}$, that is, the marginal cost of the manager's effort increases quickly. Moreover, there is an upper limit for the accuracy of the information received by the manager.

The manager has a constant absolute risk aversion (CARA) utility function with an absolute risk aversion coefficient r:

$$U_A\left(\tilde{W}_A\right) = -\exp\left\{-r\tilde{W}_A + V(r,\rho)\right\} \tag{1}$$

where $\tilde{W}_A = \beta_0 + \beta\theta(y)\tilde{x}$ is the wealth of the manager at the end of the period. The manager has no initial wealth; all her wealth comes from compensation for managing the investment.

3. Manager's optimal effort without a risk constraint

To study the impact of risk constraint on performance-based delegated portfolio management incentive contracts, we first refer to the studies by Stoughton (1993) and Gomez and Sharma (2006), in which the manager's utility and optimal effort level are studied without a risk constraint. Their results can be used as a benchmark for the results in this paper. With the model outlined in Section 2, the decision faced by the investor is:

$$\max_{\beta_0,\beta} E\left[U_B\left(\tilde{W}_B\right)\right]$$

$$s.t.$$

$$\tilde{W}_A = \beta_0 + \beta\tilde{W} \tag{2}$$

$$\tilde{W}_B = \tilde{W} - \left(\beta_0 + \beta\tilde{W}\right) \tag{3}$$

$$\tilde{W} = \theta(y)\tilde{x} \tag{4}$$

$$\theta(y) \in \arg\max E\left[U_A\left(\tilde{W}_A\right)|\tilde{y} = y\right] \tag{5}$$

$$\rho \in \arg\max E\left[U_A\left(\tilde{W}_A\right)\right] \tag{6}$$

$$E\left[U_A\left(\tilde{W}_A\right)\right] \geq -1 \tag{7}$$

where $E\left[U_B\left(\tilde{W}_B\right)\right]$ is the expected utility of the investor, Eqs. (2) and (3) are the end-of-period wealth of the manager and the investor respectively; Eq. (4) is the return of the manager's portfolio, where $\theta(y)$ is the proportion invested in the risky asset. Constraint (5) is the optimal asset selection constraint of the manager, and Eqs. (6) and (7) are the manager's incentive compatibility and participation constraint respectively. Not to lose generality, the manager's reservation utility is assumed to be -1.

At the effort level ρ, the manager observes the private information $\tilde{y} = y$. The manager's optimization problem is

$$\max_{\theta} E[-\exp\{-r(\beta_0 + \beta\theta\tilde{x}) + V(r,\rho)\}|\tilde{y} = y]$$

where θ, the proportion invested in the risky asset, is the decision variable. The certainty equivalent problem to this optimization is:

$$\max_{\theta} \beta\theta\frac{\rho}{1+\rho}y - V(r,\rho) - \frac{r}{2}(\beta\theta)^2\frac{1}{1+\rho}\cdot$$

Its first order condition is

$$\frac{\rho}{1+\rho}y - r\theta\frac{\beta}{1+\rho} = 0,$$

so the optimal proportion invested in the risky asset by the manager is

$$\theta(y) = \frac{\rho}{r\beta}y$$

To solve the optimal effort level of the manager, we need to calculate her expected utility function. Based on Eq. (1), the conditional expected utility of the manager is

$$E\left[U_A\left(\tilde{W}_A\right)\middle|\tilde{y} = y\right] = \exp\left\{-\frac{\rho^2}{2(1+\rho)}y^2\right\}\exp\{-r\beta_0 + V(r,\rho)\}\cdot$$

The unconditional expected utility of the manager is

$$E\left[U_A\left(\tilde{W}_A\right)\right] = \frac{-1}{\sqrt{1+\rho}}\exp\{-r\beta_0 + V(r,\rho)\}\cdot \tag{8}$$

Eq. (8) shows that the manager's expected utility is independent of the wealth-sharing parameter β in the contract. The expected utility maximizing effort solves the first order condition (8):

$$V'_{\rho}(r,\rho) = \frac{1}{2(1+\rho)}\cdot \tag{9}$$

Eq. (9) shows that the effort level of the manager is a function of her absolute risk aversion coefficient r, but independent of the wealth-sharing parameter β. In other words, with the absence of a risk constraint the marginal effect of β is zero — this is the well-known underinvestment in effort with a linear contract as argued by Stoughton (1993). The intuition behind this result is that without any investing constraint, the manager is always able to undo the effects of incentives by appropriate modifications of the portfolio.

4. The VaR constraint

Value at risk, or VaR, measures the maximum loss under normal market conditions. More specifically, it indicates the maximum loss at a certain confidence level over a certain period of time:

$$\mathrm{Prob}(\Delta L \leq -\mathrm{VaR}) = 1 - \alpha$$

where ΔL is the loss in the portfolio over a period of time Δt and α is the confidence level. Since VaR provides a reasonable and easy-to-understand measure of the downside risk in a portfolio, it has become the most popular choice of risk constraint by investors in delegated portfolio management since the 1990s.

Suppose the return of the portfolio follows a normal distribution with an expectation μ and a standard deviation σ, then the VaR at a confidence level α is

$$\mathrm{VaR} = Z_\alpha\sigma - \mu$$

where Z_α is the critical value under a standard normal distribution at the confidence level α.

Suppose the risk constraint in a delegated portfolio management contract is $\mathrm{VaR} \leq V_0$, where V_0 is a level of loss pre-specified by the investor. Then the risk constraint faced by the manager is:

$$Z_\alpha\sigma - \mu \leq V_0$$

As discussed earlier, if the manager accepts a delegated portfolio management contract, pays the effort ρ, receives a private signal

$\tilde{y}=y$, and the optimal proportion invested in risky asset is θ, then the manager's portfolio has an expected return of $\frac{\theta\rho}{1+\rho}y$ with a standard deviation $|\theta|\frac{1}{\sqrt{1+\rho}}$. The VaR risk constraint faced by the manager is:

$$Z_\alpha\sigma-\mu=Z_\alpha|\theta|\frac{1}{\sqrt{1+\rho}}-\frac{\theta\rho}{1+\rho}y\le V_0 \tag{10}$$

According to Hull (2008), among others, the expected return of a portfolio over a short period of time is usually much smaller than its standard deviation. So the expected return is often assumed to be zero when calculating VaR. Thus the VaR constraint (10) can be approximated by

$$Z_\alpha|\theta|\frac{1}{\sqrt{1+\rho}}\le V_0 \tag{11}$$

Let $V_\alpha=V_0/Z_\alpha$, then the constraint (11) is equivalent to

$$\begin{cases}\theta+V_\alpha\sqrt{1+\rho}\ge 0\\ \theta-V_\alpha\sqrt{1+\rho}\le 0\end{cases} \tag{12}$$

5. The manager's utility with VaR constraint

With the addition of a VaR constraint to the decision model, the optimal asset choice Eq. (5) by the manager becomes

$$\begin{aligned}&\theta(y)=\arg\max E\left[U_A\left(\tilde{W}_A\right)|\tilde{y}=y\right]\\ &s.t.\begin{cases}\theta+V_\alpha\sqrt{1+\rho}\ge 0\\ \theta-V_\alpha\sqrt{1+\rho}\le 0\end{cases}\end{aligned} \tag{13}$$

Let $\lambda_1\ge 0$ and $\lambda_2\ge 0$ be the Lagrange multipliers of restriction Eq. (12), then the Lagrange function is

$$\begin{aligned}L(\theta;\lambda_1,\lambda_2)=&\beta\theta\frac{\rho}{1+\rho}y-\frac{r}{2}(\beta\theta)^2\frac{1}{1+\rho}\\ &+\lambda_1\left(\theta+V_\alpha\sqrt{1+\rho}\right)-\lambda_2\left(\theta-V_\alpha\sqrt{1+\rho}\right)\end{aligned} \tag{14}$$

According to the Kuhn–Tucker condition, at optimum $\lambda_1(\theta+V_\alpha\sqrt{1+\rho})=0$ and $\lambda_2\left(\theta-V_\alpha\sqrt{1+\rho}\right)=0$. With the condition $\tilde{y}=y$ there are three possible solutions to the manager's optimization problem Eq. (13).

(1) When $\lambda_1\ne 0$ and $\lambda_2=0$ the upper limit of the asset choice is not binding,

$$\frac{\partial L}{\partial\theta}=\beta\frac{\rho}{1+\rho}y-r\theta\beta^2\frac{1}{1+\rho}+\lambda_1=0 \tag{15}$$

$$\frac{\partial L}{\partial\lambda_1}=\theta+V_\alpha\sqrt{1+\rho}=0 \tag{16}$$

Solving Eqs. (15) and (16) and letting $l=\frac{\sqrt{1+\rho}}{\rho}$ give

$$\lambda_1=-\frac{\rho\beta}{1+\rho}(y+V_\alpha r\beta\, l),\theta=-V_\alpha\sqrt{1+\rho}$$

(2) When $\lambda_1=0$ and $\lambda_2\ne 0$ the lower limit of the asset choice is not binding. Similarly we can get

$$\lambda_2=\frac{\rho\beta}{1+\rho}(y-V_\alpha r\beta\, l),\theta=V_\alpha\sqrt{1+\rho}$$

(3) When $\lambda_1=0$ and $\lambda_2=0$, the investment choice is not restricted by the VaR constraint, and $\theta=\frac{\rho}{r\beta}y$.

Now we study the optimal investment decision θ as a function of the private information (y) of the manager. Since $\lambda_1=-\frac{\rho\beta}{1+\rho}(y+V_\alpha r\beta\, l)>0$ is equivalent to $y<-V_\alpha r\beta\, l$ and $\lambda_2=\frac{\rho\beta}{1+\rho}(y-V_\alpha r\beta\, l)>0$ is equivalent to $y>V_\alpha r\beta\, l$, so the investment decision is

$$\theta(y)=\begin{cases}-V_\alpha\sqrt{1+\rho}, & y<-V_\alpha r\beta\, l\\ \frac{\rho y}{r\beta}, & |y|\le V_\alpha r\beta\, l\\ V_\alpha\sqrt{1+\rho}, & y>V_\alpha r\beta\, l\end{cases} \tag{17}$$

As for the conditional distribution $\tilde{x}\Big|y\sim N\left(\frac{\rho}{1+\rho}y,\frac{1}{1+\rho}\right)$, when $y<-V_\alpha r\beta\, l$,

$$\begin{aligned}&E\left[U_A\left(\tilde{W}_A\right)|\tilde{y}=y)\right]\\ &=-\exp(-r\beta_0+V(a,\rho))\exp\left\{\frac{1}{l}V_\alpha r\beta\left(y+\frac{V_\alpha r\beta\, l}{2}\right)\right\}.\end{aligned}$$

Similarly we can get the conditional expected utility of the manager when $|y|\le V_\alpha r\beta\, l$ and $y>V_\alpha r\beta\, l$. Therefore

$$\begin{aligned}E\left[U_A\left(\tilde{W}_A\right)\Big|\tilde{y}=y)\right]=&-\exp\{-r\beta_0+V(r,\rho)\}\\ &\times\begin{cases}\exp\left\{\frac{1}{l}V_\alpha r\beta\left(y+\frac{V_\alpha r\beta\, l}{2}\right)\right\}, & y<-V_\alpha r\beta\, l\\ \exp\left\{-\frac{\rho^2}{2(1+\rho)}y^2\right\}, & |y|\le V_\alpha r\beta\, l\\ \exp\left\{-\frac{1}{l}V_\alpha r\beta\left(y-\frac{V_\alpha r\beta\, l}{2}\right)\right\}, & y>V_\alpha r\beta\, l\end{cases}\end{aligned} \tag{18}$$

To study the unconditional expected utility of the manager and investigate the connection between the expected utility and the investment return sharing ratio β, we introduce a χ^2 distribution with the degree of freedom of 1. Its cumulative density function (CDF) is $\Phi(x)=\int_0^x\varphi(t)dt$ with the probability density function (PDF) as

$$\varphi(t)=\begin{cases}\frac{1}{\sqrt{2\pi}}t^{-1/2}e^{-t/2}, & t>0\\ 0, & \text{otherwise}\end{cases}$$

Since $\tilde{y}\sim N(0,1+\rho^{-1})$ and let $k=V_\alpha^2r^2\beta^2$, then from Eq. (18) the unconditional expected utility of the manager is

$$E\left[U_A\left(\tilde{W}_A\right)\right]=-\exp\{-r\beta_0+V(r,\rho)\}\times h(\rho|\beta) \tag{19}$$

where

$$\begin{aligned}h(\rho|\beta)=&\frac{1}{\sqrt{1+\rho}}\Phi\left[\frac{k(1+\rho)}{\rho}\right]\\ &+\exp\left[\frac{k(1+\rho)}{2}\right]\left(1-\Phi\left[\frac{k(1+\rho)^2}{\rho}\right]\right).\end{aligned} \tag{20}$$

$h(\rho|\beta)$ indicates that the right hand side of Eq. (20) is a function of ρ and β, and ρ is a function of β. When in the VaR constraint $V_\alpha\to+\infty$ or the risk aversion factor of the manager $r\to+\infty, h(\rho|\beta)\to\frac{1}{\sqrt{1+\rho}}$. The manager's expected utility Eq. (8) in the absence of a risk constraint is the limit of Eq. (19).

Take partial differential with respect to β in Eq. (20), and notice that

$$\frac{1}{\sqrt{1+\rho}}\varphi\left[\frac{k(1+\rho)}{\rho}\right]=\exp\left\{\frac{k(1+\rho)}{2}\right\}\varphi\left[\frac{k(1+\rho)^2}{\rho}\right].$$

Then some further calculations give

$$\frac{\partial}{\partial\beta}h\left(\rho\middle|\beta\right)=-2V_\alpha^2r^2(1+\rho)\beta\left\{\varphi\left[\frac{k(1+\rho)^2}{\rho}\right]-\frac{1}{2}\left(1-\Phi\left[\frac{k(1+\rho)^2}{\rho}\right]\right)\right\}\exp\left[\frac{k(1+\rho)}{2}\right]$$

Since when $0<x<+\infty, \varphi(x)-\frac{1}{2}(1-\Phi(x))=\frac{1}{2}\int_x^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-t/2}t^{-3/2}dt>0$, then when $x=\frac{k(1+\rho)^2}{\rho}$,

$$\varphi\left[\frac{k(1+\rho)^2}{\rho}\right]-\frac{1}{2}\left(1-\Phi\left[\frac{k(1+\rho)^2}{\rho}\right]\right)>0 \tag{21}$$

so

$$\frac{\partial}{\partial\beta}h(\rho|\beta)<0$$

Combining Eqs. (19) and (21) gives

$$\frac{\partial}{\partial\beta}\left(E\left[U_A\left(\tilde{W}_A\right)\right]\right)>0 \tag{22}$$

Therefore we reach the following result:

Proposition 1. *Under a VaR constraint the manager's expected utility increases with the return sharing ratio β.*

Proposition 1 shows that the introduction of a VaR risk constraint changes the connection between the return sharing ratio and the manager's utility. When $|y|\le V_\alpha r\beta l$ the manager's investment decision is not affected by the risk constraint. The marginal effect of β is zero, that is, the contract has no impact upon the optimal effort level of the manager (this is the no-incentive result in Stoughton, 1993). When $|y|>V_\alpha r\beta l$, however, the manager's investment decision is restricted by the risk constraint. As the return sharing ratio β or the manager's absolute risk aversion coefficient r increases, the risk constraint has less effect on the portfolio investment decision.

6. Incentive impact of VaR constraint

Now we study the manager's effort level under a VaR constraint. Taking partial differentials with respect to effort level ρ in Eq. (20) gives

$$\frac{\partial}{\partial\rho}h\left(\rho|\beta\right)=-\frac{1}{2}\left(\frac{1}{1+\rho}\right)^{3/2}\Phi\left[\frac{k(1+\rho)}{\rho}\right]-k\exp\left[\frac{k(1+\rho)}{2}\right]\left\{\varphi\left[\frac{k(1+\rho)^2}{\rho}\right]-\frac{1}{2}\left(1-\Phi\left[\frac{k(1+\rho)^2}{\rho}\right]\right)\right\}. \tag{23}$$

From Eq. (21) we can determine

$$\frac{\partial}{\partial\rho}h(\rho|\beta)<0. \tag{24}$$

As for the manager's expected utility Eq. (19), the equivalent objective function for the manager to choose her optimal effort level is

$$\max_\rho -\exp\{V(r,\rho)\}h(\rho|\beta) \tag{25}$$

By the first order condition

$$V'_\rho(r,\rho)=-\frac{h'_\rho\left(\rho\middle|\beta\right)}{h(\rho|\beta)} \tag{26}$$

Result Eq. (26) shows that the manager's optimal effort level ρ is a function of the return sharing ratio β. As in the VaR constraint $V_\alpha\to+\infty$ or the manager's risk aversion coefficient $r\to+\infty$, $\frac{\partial}{\partial\rho}h(\rho|\beta)\to-\frac{1}{2}(1+\rho)^{-\frac{3}{2}}$ and $V'_\rho(r,\rho)\to\frac{1}{2(1+\rho)}$. In other words, Eq. (9) is the limit of Eq. (26) — the situation without any restriction is a special case and the limit of the case with the VaR constraint. In Eq. (9) the return sharing ratio β is irrelevant to $V'_\rho(r,\rho)$.

To further investigate the impact of a VaR constraint on the manager's effort level, we compare the marginal cost of the manager's effort with and without a VaR constraint. From Eqs. (9) and (26),

$$-\frac{h'_\rho\left(\rho\middle|\beta\right)}{h\left(\rho\middle|\beta\right)}-\frac{1}{2(1+\rho)}=\frac{\exp\left\{\frac{k(1+\rho)}{2}\right\}G(k,\rho)}{\left(\frac{1}{1+\rho}\right)^{1/2}\Phi\left[\frac{k(1+\rho)}{\rho}\right]+\exp\left\{\frac{k(1+\rho)}{2}\right\}\left(1-\Phi\left[\frac{k(1+\rho)^2}{\rho}\right]\right)\}} \tag{27}$$

where

$$G(k,\rho)=k\left(\Phi\left[\frac{k(1+\rho)^2}{\rho}\right]-\frac{1}{2}\left(1-\Phi\left[\frac{k(1+\rho)^2}{\rho}\right]\right)\right)-\frac{1}{2(1+\rho)}\left(1-\Phi\left[\frac{k(1+\rho)^2}{\rho}\right]\right)$$

According to the CDF and PDF of the χ^2 distribution with a degree of freedom of 1,

$$G(k,\rho)=\int_{\frac{k(1+\rho)^2}{\rho}}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-t/2}t^{-1/2}\left[\frac{k}{2}\frac{1}{t}-\frac{1}{2(1+\rho)}\right]dt$$

Since $t\in\left[\frac{k(1+\rho)^2}{\rho},+\infty\right)$, we get $\frac{k}{2}\frac{1}{t}-\frac{1}{2(1+\rho)}\le-\frac{1}{2(1+\rho)^2}<0$. Because the integrand is negative in the definite integral $G(k,\rho)$, $G(k,\rho)<0$. Therefore

$$-\frac{h'_\rho(\rho|\beta)}{h(\rho|\beta)}-\frac{1}{2(1+\rho)}<0. \tag{28}$$

Result Eq. (28) shows that the marginal cost of the manager's effort with a VaR constraint is lower than the marginal cost without a VaR constraint. Based on this observation and the assumption $V''_{\rho\rho}(r,\rho)\ge 0$ (the marginal cost of effort increases with effort level), we have the following result:

Proposition 2. *Under a VaR constraint the marginal cost of the manager's effort is a function of the return sharing ratio, so a linear contract has incentive effects on the manager. However, the manager's effort level with a VaR constraint is lower than that without such a constraint.*

An explanation for Proposition 2 is that, from Eq. (17), with a VaR constraint there exists a certain probability ($|y|>V_\alpha r\beta l$) such that the manager cannot construct an optimal portfolio based on the private information received (in such situations the portfolio is determined by the risk constraint). Since $\partial l/\partial\rho<0$, by exerting more effort the manager could actually “enhance” the restriction induced by VaR. In response, the manager lowers her effort in gathering better information. Proposition 2 tells us that a risk constraint increases the moral hazard between the investor and the manager.

Now we further analyze the relationship between the manager's effort level ρ and the return sharing ratio β. As we have learned, ρ and β satisfy the implicit function Eq. (26). Taking partial differential with respect to β at both sides of Eq. (26) gives

$$\frac{\partial\rho}{\partial\beta}=-\frac{V'_\rho(r,\rho)\frac{\partial h}{\partial\beta}+\frac{\partial}{\partial\beta}\left(\frac{\partial h}{\partial\rho}\right)}{V''_{\rho\rho}(r,\rho)h(\rho)+V'_\rho(r,\rho)\frac{\partial h}{\partial\rho}+\frac{\partial^2h}{\partial\rho^2}} \tag{29}$$

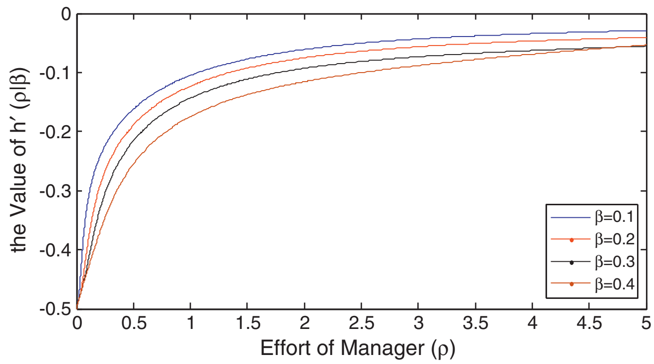

Fig. 1. $h'_\rho(\rho|\beta)$vs.$\rho(V_\alpha=0.02, r=0.1)$.

Unfortunately, it is very difficult to judge directly the signs of $\frac{\partial^2 h}{\partial \rho^2}$ and $\frac{\partial}{\partial \beta}\left(\frac{\partial h}{\partial \rho}\right)$, so we have to turn for help to numerical analyses, as in Figs. 1 and 2.

Figs. 1 and 2 show that $\frac{\partial h}{\partial \rho}$ increases with the manager's effort level, i.e., $\frac{\partial^2 h}{\partial \rho^2}>0$, and $\frac{\partial h}{\partial \rho}$ decreases with β, i.e., $\frac{\partial}{\partial \beta}\left(\frac{\partial h}{\partial \rho}\right)<0$. Some further analytical work (see the Appendix for details) shows $\frac{\partial \rho}{\partial \beta}>0$. Now we reach the following result:

Proposition 3. *Under a VaR constraint the manager's effort level increases with the return sharing ratioβ — a linear contract can induce the manager to work hard.*

Proposition 3 shows that, with a VaR constraint, a linear contract can not only make the investor and the manager share the risk but also induce the manager to work hard. This result is in contrast with those in Stoughton (1993) and Admati and Pleiderer (1997). Proposition 2 shows that the introduction of a VaR constraint decreases the manager's effort level. However, if the manager is provided with a stronger incentive — a higher return sharing ratio, the manager will work harder to acquire a private signal, which increases the accuracy of her private information. The higher the accuracy of the private information, the lower the variance of the portfolio managed by the manager. The smaller the impact of the VaR constraint on portfolio building, the more freedom for the manager to choose assets. Remember in Eq. (17) when $|y|>V_\alpha r\beta l$ the manager's investment strategy is restricted by the VaR constraint. However, a higher return sharing ratio β can relax this restriction, thus increasing the freedom (*i.e.*, portfolio opportunity set) for the manager. Also, the impact of the VaR is inversely related to the manager's risk aversion: *i.e.*, the larger is r the smaller is the effect of VaR on the manager's investment strategy and effort decision. In short, with a VaR risk constraint, a higher return sharing ratio can induce the manager to work harder to gain better private information, thereby partially undoing the effects imposed by the VaR constraint in the first place, which benefits both the manager and the investor. This result is consistent with the argument in Dybvig et al. (2010), which asserts that trading restrictions are an essential part of an optimal contract because they prevent the manager from undoing the incentive effects of performance-based fees.

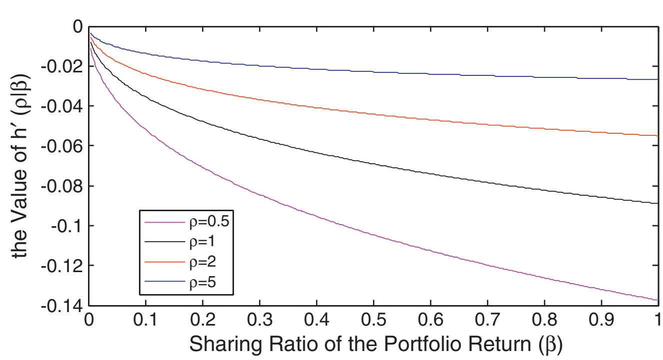

Fig. 2. $h'_\rho(\rho|\beta)$vs.β $(V_\alpha=0.02, r=0.1)$.

Together Propositions 2 and 3 show that, in the presence of a VaR constraint, the return sharing ratio in a linear contract plays an additional role over risk-sharing. As in most moral hazard problems, efficiency in risk allocation has to be traded off against effort inducement.

7. Conclusion

This paper investigates the impact of a VaR constraint on a linear contract for delegated portfolio management within a principal-agent framework. Under a VaR constraint, increasing the return sharing ratio in the contract can increase the freedom for the manager when building the portfolio, and therefore partially lessen the impact of the risk constraint. A risk-averse manager's expected utility and optimal effort level increase with the return sharing ratio, so a linear contract can not only make the investor and the manager share risks efficiently, but also motivate the manager to gain high-quality private information, which is mutually beneficial to the manager and the investor. However, the manager's effort level is lower with a VaR constraint than without. This result shows that although a VaR constraint can limit the risk-taking behavior of the manager, it also lowers the manager's effort level, thus raising the moral hazard problem.

Future research may investigate the optimal contract under a VaR constraint. Our preliminary work, however, indicates that such an optimal contract may not be available in a closed form.

Appendix: Derivation of Proposition 3

From Eq. (21) we have determined $\frac{\partial h}{\partial \beta}<0$, so

$$V'_\rho(r,\rho)\frac{\partial h}{\partial \beta}+\frac{\partial}{\partial \beta}\left(\frac{\partial h}{\partial \rho}\right)<0 \tag{30}$$

In Eq. (29) let $\frac{\partial h}{\partial \rho}=h'_\rho(\rho|\beta)$ and $\frac{\partial^2 h}{\partial \rho^2}=h''_{\rho\rho}(\rho|\beta)$, then the denominator at the right hand side of Eq. (29) can be expressed as

$$h(\rho|\beta)\left[V''_{\rho\rho}(r,\rho)+V'_\rho(r,\rho)\frac{h'_\rho(\rho|\beta)}{h(\rho|\beta)}+\frac{h''_{\rho\rho}(\rho|\beta)}{h(\rho|\beta)}\right] \tag{31}$$

From the assumption $\frac{V''_{\rho\rho}(r,\rho)\rho}{V'_\rho(r,\rho)}>\frac{\rho}{1+\rho}$ we can get $V''_{\rho\rho}(r,\rho)>\frac{1}{1+\rho}V'_\rho(r,\rho)$. By Eq. (26), part of Eq. (31) leads to

$$\begin{aligned}&V''_{\rho\rho}(r,\rho)+V'_\rho(r,\rho)\frac{h'_\rho(\rho|\beta)}{h(\rho|\beta)}=V''_{\rho\rho}(r,\rho)-\left(V'_\rho(r,\rho)\right)^2\\&>V'_\rho(r,\rho)\left[\frac{1}{1+\rho}-V'_\rho(r,\rho)\right]\end{aligned} \tag{32}$$

From Eqs. (26) and (28), $0<V'_\rho(r,\rho)<\frac{1}{2(1+\rho)}$, then it follows in Eq. (32) that,

$$\begin{aligned}&V'_\rho(r,\rho)\left[\frac{1}{1+\rho}-V'_\rho(r,\rho)\right]>V'_\rho(r,\rho)\left[\frac{1}{1+\rho}-\frac{1}{2(1+\rho)}\right]\\&=\frac{1}{2(1+\rho)}V'_\rho(r,\rho)>0\end{aligned} \tag{33}$$

So Eq. (33) gives $V''_{\rho\rho}(r,\rho)+V'_{\rho}(r,\rho)\dfrac{h'_{\rho}(\rho|\beta)}{h(\rho|\beta)}>0$. From Eq. (20) and the observation in Fig. 1, $\frac{h''_{\rho\rho}(\rho|\beta)}{h(\rho|\beta)}>0$, so we determine that the expression in Eq. (31) is positive. Then putting it together with $\frac{\partial h}{\partial\beta}<0$ and $\frac{\partial}{\partial\beta}\left(\frac{\partial h}{\partial\rho}\right)<0$ gives,

$$\frac{\partial\rho}{\partial\beta}=-\frac{V'_{\rho}(r,\rho)\frac{\partial h}{\partial\beta}+\frac{\partial}{\partial\beta}\left(\frac{\partial h}{\partial\rho}\right)}{V''_{\rho\rho}(r,\rho)h(\rho)+V'_{\rho}(r,\rho)\frac{\partial h}{\partial\rho}+\frac{\partial^2 h}{\partial\rho^2}}>0 \tag{34}$$

that is, $\frac{\partial\rho}{\partial\beta}>0$, which arrives at Proposition 3.

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